On the controllability of a special class of coupled wave systems

Jingrui NIU

Sorbonne Université

ICIAM, Tokyo, 22 August 2023

Based on joint work with Pierre Lissy

General settings

We consider Ω to be a bounded domain $\mathbb{R}^d$ with smooth boundary. Let $\omega \subset \Omega$ be a subdomain. We aim to investigate the exact/null controllability of the following type of coupled wave systems:

$$\begin{cases} (\partial_t^2 - D\Delta)U + AU &= BF\mathbf{1}_\omega & \text{in } (0,T)\times\Omega, \\ U &= 0 & \text{on } (0,T)\times\partial\Omega, \\ (U,\partial_t U)|_{t=0} &= (U^0,U^1) & \text{in } \Omega, \end{cases} \tag{GCW}$$

with here

$$D = \mathrm{diag}(d_1,\cdots,d_n)_{n\times n}, A \in \mathcal{M}_{n\times n}(\mathbb{R}), \text{ and } B \in \mathcal{M}_{n\times m}(\mathbb{R})(m \le n)$$

and $F = (F_1,\cdots,F_m)$ is our control.

A review of literature

There is a large literature on the controllability of the wave equations.

- For a single wave equation: Bardos-Lebeau-Rauch '92, Lions '88, Baudouin-De Buhan-Ervedoza '13 $\cdots$
- For wave systems in same speed: Alabau-Boussouira '03,'13, Alabau-Boussouira-Léautaud '13, Liard-Lissy '17, Lissy-Zuazua '19, Cui-Laurent-Wang '20
- For wave systems in different speeds: Dehman-Le Rousseau-Léautaud '14, Lissy-Zuazua '19, N '21

Some links with other problems

- Ammar-Khodja-Benabdallah-Dupaix-González-Burgos '09 (parabolic)
- Li-Rao '12, '13 (synchronisation of waves)

Goal

Under the geometric assumptions+ algebraic conditions, the system GCW is exactly controllable.

Geometric Control Condition

Let p_g be the principal symbol of the operator $\partial_t^2 - \Delta_g$.

Definition

For $\omega \subset \Omega$ and $T > 0$, we shall say that the pair (ω, T, p_g) satisfies GCC if every general bicharacteristic of p_g meets ω in a time $t < T$.

This is a very important condition when one considers the control of waves. One can refer Rauch-Taylor 74', Bardos-Lebeau-Rauch 88',92', Burq-Gérard 97',$\cdots$

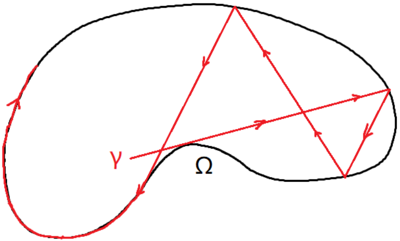

Figure: General bicharacteristics

Kalman conditions

Definition (Kalman operator)

Let m, n be two positive integers. Assume that $X \in \mathcal{M}_n(\mathbb{R})$ *and* $Y \in \mathcal{M}_{n,m}(\mathbb{R})$. *Moreover, let D be a diagonal matrix. Then, the Kalman operator associated with* $(-D\Delta + X, Y)$ *is the matrix operator* $\mathscr{K} = [-D\Delta + X|Y] = [(D\Delta + X)^{n-1}Y|\cdots|(D\Delta + X)Y|Y] : D(\mathscr{K}) \subset (L^2)^{nm} \to (L^2)^n)$.

Definition (Operator Kalman rank condition)

We say that the Kalman operator $\mathscr{K}$ *satisfies the operator Kalman rank condition if* $Ker(\mathscr{K}^*) = \{0\}$.

Microlocal defect measure-1

Based on Gérard-Leichtnam 93' and Burq 97'. Let $(u^k)_{k\in\mathbb{N}}$ be a bounded sequence in $L^2_{loc}(\mathbb{R}^+; L^2(\Omega))$, converging weakly to 0 and such that

$$\begin{cases}(\partial_t^2 - \Delta)u^k = 0,\\ u^k|_{\partial\Omega} = 0.\end{cases} \tag{1}$$

Let $\underline{u}^k$ be the extension by 0 across the boundary of Ω. Then the sequence $\underline{u}^k$ is bounded in $L^2_{loc}(\mathbb{R}_t; L^2(\mathbb{R}^d))$. Let $\underline{\mathcal{A}}$ be the space of classical pseudo-differential operators of order 0 with compact support in $\mathbb{R}^+ \times \mathbb{R}^d$

Proposition

There exists a subsequence of $(\underline{u}^k)$ (still noted by $(\underline{u}^k)$) and $\underline{\mu} \in \underline{\mathcal{M}}^+$ such that

$$\forall A \in \underline{\mathcal{A}}, \quad \lim_{k\to\infty} (A\underline{u}^k, \underline{u}^k)_{L^2} = \langle \underline{\mu}, \sigma(A)\rangle, \tag{2}$$

where $\sigma(A)$ is the principal symbol of the operator A (which is a smooth function, homogeneous of order 0 in the variable ξ, i.e. a function on $S^*((\mathbb{R}^+ \times \mathbb{R}^d))$.

Microlocal defect measure-2

For the microlocal defect measure $\underline{\mu}$ defined before, we have the following properties.

- $\mathrm{supp}(\underline{\mu}) \subset Char(P) = \{(t, x, \tau, \xi); x \in \overline{M}, \tau^2 = |\xi|_x^2\}$.
- $\underline{\mu}$ is invariant along the generalized bicharacteristic flow.

A simple model

$$\begin{cases} (\partial_t^2 - \Delta)u_1 + u_2 & = 0 \quad \text{in } (0,T)\times\Omega, \\ (\partial_t^2 - 2\Delta)u_2 + u_3 & = 0 \quad \text{in } (0,T)\times\Omega, \\ (\partial_t^2 - 2\Delta)u_3 & = f\mathbf{1}_\omega \quad \text{in } (0,T)\times\Omega, \end{cases} \tag{M2}$$

with the Dirichlet boundary condition and some initial data. This system has the following features:

- f is only acting directly on u_3,
- u_2 and u_3 are coupled via a weak coupling (lower order),
- u_1 and u_2 are coupled via a very weak coupling (lower order+different speed).

$\Rightarrow$ Compatibility conditions.

Question

- What are the compatibility conditions for this system (M2)?
- Is it controllable?

On regularity of the system (M2)

For this example system, we begin with zero initial conditions.

$(u_1, u_2, u_3) \in H^4 \times H^2 \times H^1$

In fact, it is classic to prove that

$$u_3 \in C^0([0,T], H^1) \cap C^1([0,T], L^2),$$
$$u_2 \in C^0([0,T], H^2) \cap C^1([0,T], H^1).$$

For u_1, $\Box_1 u_1 = -u_2$, which implies that $\Box_2 \Box_1 u_1 = -\Box_2 u_2 = u_3$. Hence, we obtain that $\Box_2 u_1 \in C^0 H^2 \cap C^1 H^1$. And we already know that $\Box_1 u_1 = -u_2 \in C^0 H^2 \cap C^1 H^1$. Take the difference, we obtain that $\Delta u_1 \in C^0 H^2 \cap C^1 H^1$ which implies that $u_1 \in C^0 H^4 \cap C^1 H^3$.

Compatibility conditions

$(-\Delta)^2 u_1 + \Delta u_2 \in H_0^1$.

Compatibility conditions

We introduce a transform $\mathcal{S}$ by

$$\mathcal{S}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} D_t^3 u_1, \\ D_t u_2, \\ u_3. \end{pmatrix}.$$

Moreover, (v_1, v_2, v_3) satisfies the following system:

$$\begin{cases} \Box_1 v_1 + D_t^2 v_2 = 0 \text{ in } (0, T) \times \Omega, \\ \Box_2 v_2 + D_t v_3 = 0 \text{ in } (0, T) \times \Omega, \\ \Box_2 v_3 = f \text{ in } (0, T) \times \Omega. \end{cases} \tag{M2v}$$

Using the identity $-D_t^2 = 2\Box_1 - \Box_2$, we have that $\Box_1(v_1 - 2v_2) - D_t v_3 = 0$. Hence, $\Box_1(D_t v_1 - 2D_t v_2 + 2v_3) = f$. However, we know that $D_t v_1 - 2D_t v_2 + 2v_3 = (-\Delta)^2 u_1 + \Delta u_2 + u_3$, which implies that $(-\Delta)^2 u_1 + \Delta u_2 \in H_0^1$.

A wave system coupled with different speeds

To generalize the previous model, we deal with the controllability of the following type of coupled wave systems:

$$\begin{cases} (\partial_t^2 - D\Delta)U + AU &= \hat{b} f \mathbf{1}_\omega & \text{in } (0,T)\times\Omega, \\ U &= 0 & \text{on } (0,T)\times\partial\Omega, \\ (U, \partial_t U)|_{t=0} &= (U^0, U^1) & \text{in } \Omega, \end{cases} \qquad \text{(CWS)}$$

with here

$$D = \begin{pmatrix} d_1 Id_{n_1} & 0 \\ 0 & d_2 Id_{n_2} \end{pmatrix}_{n\times n}, A = \begin{pmatrix} 0 & A_1 \\ 0 & A_2 \end{pmatrix}_{n\times n}, \text{ and } \hat{b} = \begin{pmatrix} 0 \\ b \end{pmatrix}_{n\times 1},$$

where $n = n_1 + n_2$ and $d_1 \neq d_2$. $A_1 \in \mathcal{M}_{n_1,n_2}(\mathbb{R})$ and $A_2 \in \mathcal{M}_{n_2}(\mathbb{R})$ are two given coupling matrices and $b \in \mathbb{R}^{n_2}$.

Kalman rank condition

Proposition

We denote by $\mathcal{K} = [-D\Delta + A|\hat{b}]$ *the Kalman operator associated with System* (CWS). *Then,* $Ker(\mathcal{K}^*) = \{0\}$ *is equivalent to satisfying all the following conditions:*

1. $n_1 = 1$;
2. (A_2, b) *satisfies the usual Kalman rank condition;*
3. *Assume that* $A_1 = \alpha = (\alpha_1, \cdots, \alpha_{n_2})$. *Then* $\forall \lambda \in \sigma(-\Delta)$, α *satisfies*

$$\alpha \left(\sum_{k=0}^{n_2-2} (d_1 - d_2)^k \lambda^k \sum_{j=k+1}^{n_2} a_j A_2^{j-1-k} + (d_1 - d_2)^{n_2-1} \lambda^{n_2-1} Id_{n_2} \right) \widehat{b} \neq 0, \tag{KC}$$

where $(a_j)_{0 \le j \le n_2}$ *are the coefficients of the the characteristic polynomial of the matrix* A_2, *i.e.* $\chi(X) = X^{n_2} + \sum_{j=0}^{n_2-1} a_j X^j$, *with the convention that* $a_{n_2} = 1$.

Theorem

Given $T > 0$, suppose that:

1. (ω, T, p_{d_i}) *satisfies GCC, $i = 1, 2$.*
2. *Compatibility conditions.*
3. *The Kalman operator $\mathcal{K} = [-D\Delta + A|\hat{b}]$ satisfies the operator Kalman rank condition, i.e. $Ker(\mathcal{K}^*) = \{0\}$.*

Then the system (CWS) *is exactly controllable.*

Remark

As for compatibility conditions, for example, in the simple model (M2), $(u_1, u_2, u_3) \in H^4 \times H^2 \times H^1$, *we have*

$$(-\Delta)^2 u_1 + \Delta u_2 \in H_0^1.$$

Outline for the proof

We prove the above theorem within three steps.

1. At first, we simplify the system (CWS), using a Brunovskỳ normal form. Based on the equivalent Kalman condition, we prove the exact controllability for the simplified system.
2. In the second step, we use iteration schemes to obtain the compatibility conditions associated with the coupling structure. Therefore, we prepare the appropriate state spaces.
3. In the final step, we use HUM to derive the observability inequality and then follow the compactness-uniqueness procedure. At last, the unique continuation property is given by the Kalman rank condition.

Thank you for your attention!