Small-time controllability of KdV equations

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Korteweg-de Vries (KdV) model

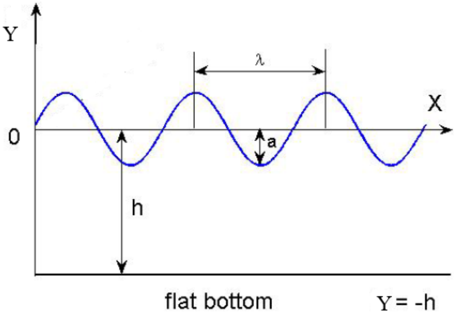


Figure: Shallow water model.

Figure: Recreation of a solitary wave on a canal by Heriot-Watt University.

In terms of the physical parameters, the KdV equation reads

$$\partial_t y + \frac{1}{2} h^2 \sqrt{gh} \left(\frac{1}{3} - \frac{\mathcal{T}}{\rho g h^2} \right) \partial_x^3 y + \sqrt{gh} \partial_x y + \frac{3}{2} \frac{\sqrt{gh}}{h} y \partial_x y = 0.$$

An open problem

Consider the KdV equation on interval

$$
\begin{aligned}
\partial_t y + \partial_x^3 y + \partial_x y + y\partial_x y = 0, \text{ in } (0,T)\times(0,L),\\
y(t,0) = y(t,L) = 0,\ \partial_x y(t,L) = u(t), \text{ in } (0,T),\\
y(0,x) = y_0(x), y(T,x) = y_1(x), \text{ in } (0,L).
\end{aligned}
$$

- Let *critical lengths set*

$$
\mathcal{N} := \{2\pi\sqrt{\frac{k^2+kl+l^2}{3}} : k,l \in \mathbb{N}^*\}
$$

Rosier (1997): the linearized system is controllable for any time $\Leftrightarrow L \notin \mathcal{N}$.

Longstanding problem

Is the KdV equation small-time locally controllable for all $L \in \mathcal{N}$?

$$L \in \mathcal{N} := \{2\pi\sqrt{\frac{k^2 + kl + l^2}{3}} : k, l \in \mathbb{N}^*\}$$

- Rosier (1997): the linear system is not controllable for any time;
- Coron–Crépeau (2003): nonlinear system is small-time locally controllable, provided that $k = l$ is the only solution pair;
- Cerpa (2007), Cerpa–Crépeau (2009): large-time locally controllable for all critical lengths;
- Coron–Koenig–Nguyen (2020): not small-time locally controllable if $2k + l \notin 3\mathbb{N}^*$;

A complete answer (N.–Xiang, 2025)

The system is not small-time locally controllable if $2k + l \in 3\mathbb{N}^*$ and $k \neq l$.

Outline of the presentation

Linear result: $\mathcal{N}$, M, and H

By Rosier (1997), *critical lengths set* $\mathcal{N} := \{2\pi\sqrt{\frac{k^2+kl+l^2}{3}} : k, l \in \mathbb{N}^*\}$. For the linearized KdV system,

- If $L \notin \mathcal{N}$, the linearized system is controllable for any $T > 0$;
- If $L \in \mathcal{N}$, the linearized system is controllable for any $T > 0$. $L^2(0, L) = H \oplus M$.
  H: *reachable subspace*
  M: *unreachable subspace*

$$M := \mathrm{Span}_{\mathbb{R}}\{\Re\varphi_\lambda, \Im\varphi_\lambda\},$$

where φ_λ solves:

$$\varphi_\lambda''' + \varphi_\lambda' + \mathrm{i}\lambda\varphi_\lambda = 0,$$
$$\varphi_\lambda(0) = \varphi_\lambda(L) = \varphi_\lambda'(0) = \varphi_\lambda'(L) = 0.$$

What about the nonlinear system?

A first nonlinear result for $L \in \mathcal{N}$

A significant step by Coron–Crépeau (2003).

Case $\dim M = 1$

The nonlinear system is small-time locally controllable for the critical lengths such that $\dim M = 1$.

Note: in this case, the linearized system is uncontrollable!

This case contains infinitely many critical lengths:

$$\{L = 2k\pi : \nexists (m, n) \text{ such that } m^2 + n^2 + mn = 3k^2 \text{ and } m \neq n.\}$$

Example: For $L = 2\pi \in \mathcal{N}$, $M = \mathbb{R}(1 - \cos x)$ and $\dim M = 1$.

Power series expansion method

Idea: decompose solutions y and search controls u in the form

$$y = \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3 + \cdots ,$$
$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots .$$

Thus

$$\begin{cases} \partial_t y_1 + \partial_x^3 y_1 + \partial_x y_1 = 0, \\ \partial_x y_1(t, L) = u_1(t). \end{cases}$$
$$\begin{cases} \partial_t y_2 + \partial_x^3 y_2 + \partial_x y_2 = -y_1 \partial_x y_1, \\ \partial_x y_2(t, L) = u_2(t). \end{cases}$$
$$\vdots$$

Fix initial states $y_1|_{t=0} = y_2|_{t=0} = 0$. Find u_1 and u_2 such that the final states satisfy $y_1|_{t=T} = 0$ and the projection of $y_2|_{t=T}$ on M is a given (nonzero) element in M.

$$y_1|_{t=0} = 0 \overset{u_1}{\rightsquigarrow} y_1|_{t=T} = 0,$$
$$y_2|_{t=0} = 0 \overset{u_2}{\rightsquigarrow} y_2|_{t=T} \in M.$$

A key quantity Q_M

Let $(\varphi, \mathrm{i}p)$ be an eigenmode in M:

$$\varphi''' + \varphi' + \mathrm{i}p\varphi = 0,$$
$$\varphi(0) = \varphi(L) = \varphi'(0) = \varphi'(L) = 0.$$

A key quantity associated with the projection on M:

$$Q_M(\varphi; y) := \int_0^{\infty} \int_0^L |y_1(t,x)|^2 e^{-\mathrm{i}pt} \varphi'(x) dx dt.$$

Vanishing of Q_M (Coron–Crépeau 2003)

Let $L = 2k\pi$. Then $Q_M(1 - \cos x; y_1) = \int_0^T \int_0^L y_1(t,x)^2 \sin x dx dt \equiv 0$.

- $M \neq \emptyset$ implies $y \sim \varepsilon y_1$ is not controllable.
- $Q_M \equiv 0$ implies $y \sim \varepsilon y_1 + \varepsilon^2 y_2$ is still not controllable...
- Further consider $y \sim \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3$: the small-time controllability.

More complicated cases

- For $\dim M = 2$, in 2007, following the idea of Coron–Crépeau (2003), Cerpa adapted the power series expansion method to prove a *large-time local controllability* result.
- Following a similar approach, Cerpa–Crépeau (2009) proved a large-time locally controllable for all critical lengths

Case $\dim M = 2$ (Cerpa 2007)

Q_M is not identically 0.

Due to this observation, he showed that the second order approximated system can arrive at a certain direction $\varphi_0 \in M$ at any short time.

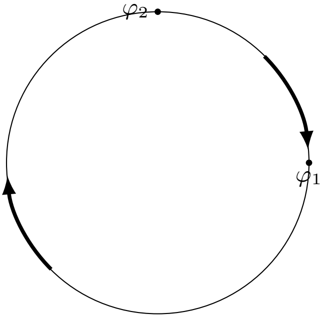
Then, the large-time controllability is fulfilled by a rotation process. While the rotation from φ_0 to any direction $e^{\mathrm{i}pt}\varphi_0$ takes a time $T \geq \frac{\pi}{p}$.

Rotation process

Let $M = \mathrm{Span}_{\mathbb{R}}\{\varphi_1, \varphi_2\}$ with $\varphi = \varphi_1 + \mathrm{i}\varphi_2$ satisfying:

$$\varphi''' + \varphi' + \mathrm{i}p\varphi = 0,$$
$$\varphi(0) = \varphi(L) = \varphi'(0) = \varphi'(L) = 0.$$

Then $\dim M = 2$ and one notices that the solution y to KdV system projects on M verifies a rotation via



$$\begin{cases} \frac{d}{dt}(y(t), \varphi_1)_{L^2(0,L)} = -p(y(t), \varphi_2)_{L^2(0,L)}, \\ \frac{d}{dt}(y(t), \varphi_2)_{L^2(0,L)} = \quad p(y(t), \varphi_1)_{L^2(0,L)}, \end{cases}$$

Since the solution can reach the direction $\varphi_0 = \alpha\varphi_1 + \beta\varphi_2$ within T_0, the rotation process $\Rightarrow$ reach all states in M if $T \geq T_0 + \frac{2\pi}{p}$.

Comments on rotation process

- This rotation approach can not answer the open problem on small-time controllability for $\dim M = 2$.
- Because of this natural process, since then people do not distinguish different L such that dim$M = 2$. For example $(k, l) = (2, 1)$ and $(k, l) = (4, 1)$.
- For other cases $\dim M > 2$, it suffices to benefit on the different rotation vitesse of eigenfunctions to reach each direction in M.

Example ($\dim M = 4$)

Assume that $M = \mathrm{Span}\{\varphi_1, \varphi_2, \phi_1, \phi_2\}$.

- the state y can reach a certain direction $\varphi_0 = \alpha\varphi_1 + \beta\varphi_2 + \gamma\phi_1 + \delta\phi_2$;
- the angle velocity of φ_i is different from the velocity of ϕ_j;
- simple superposition of linear/nonlinear solutions $\Rightarrow$ reach every state in M for T large enough.

Old classification: based on the parity of dim M

Inspired by the rotation process, the following classification has been introduced.

0. $\mathcal{C} := \mathbb{R}^+ \setminus \mathcal{N}$. Then $M = \{0\}$.
1. $\mathcal{N}_1 := \{L \in \mathcal{N};\, \exists!(k,l)$ and $k = l\}$. Then $\dim M = 1$.
2. $\mathcal{N}_2 := \{L \in \mathcal{N};\, \exists!(k,l)$ and $k > l\}$. Then $\dim M = 2$.
3. $\mathcal{N}_3 := \{L \in \mathcal{N};\, \exists n \geqslant 2$ different pairs (k,l) , and $k \neq l\}$. Then $\dim M = 2n$.
4. $\mathcal{N}_4 := \{L \in \mathcal{N};\, \exists n \geqslant 2$ different pairs (k,l) , and one of them satisfies $k = l$ $\}$. Then $\dim M = 2n - 1$.

Summary: Controllability results

	Small-time controllability	Large-time controllability
$\mathcal{C}$	Rosier(1997)	Rosier(1997)
$\mathcal{N}_1$	Coron–Crépeau (2003)	Coron–Crépeau (2003)
$\mathcal{N}_2$	Partial result: Coron–Koenig–Nguyen (2020)	Cerpa (2007)
$\mathcal{N}_3$	Partial result: Coron–Koenig–Nguyen (2020)	Cerpa–Crépeau (2009)
$\mathcal{N}_4$	Unknown	Cerpa–Crépeau (2009)

Table: Control results based on the parity of dim M

Fruitful results on different topics

	Exponential stabilization	Asymptotic stability
$\mathcal{C}$	Coron–Lv (2014)	Perla-Menzala–Vasconcellos–Zuazua (2002)
$\mathcal{N}_1$	Unknown	Chu–Coron–Shang (2015) Nguyen (2021)
$\mathcal{N}_2$	Coron–Rivas–Xiang (2017)	Tang–Chu–Shang–Coron (2018) Nguyen (2021)
$\mathcal{N}_3$	Coron–Rivas–Xiang (2017)	Partial result: Nguyen (2021)
$\mathcal{N}_4$	Unknown	Unknown

Table: Other results based on the parity of dim M

A surprising negative result

The breakthrough result on small-time controllability is as follows:

Coron–Koenig–Nguyen (2020)

By adding another assumption:

every pair (k, l) must satisfy $2k + l \notin 3\mathbb{N}^*$,

small-time controllability cannot be achieved for such critical lengths.

Note: in such case, $\dim M$ must be even. But meanwhile, even in the case $\dim M = 2$ there are many critical lengths that do not satisfy such an assumption.

However, their result cannot match the old classification very well. And in their paper, they made no further comments on this condition $2k + l \notin 3\mathbb{N}^*$.

Outline of the presentation

Motivations

From a limiting perspective, what happens when $L \to \mathcal{N}$?

Question

When $L \to L_0 \in \mathcal{N}$,

- can we find $L^2(0, L) = H(L) \oplus M(L)$ such that $H(L) \sim H(L_0)$ and $M(L) \sim M(L_0)$?
- asymptotic behaviors for $H(L)$ and $M(L)$?
- influence of $H(L)$ and $M(L)$ in nonlinear case?

Briefly, M is formulated through a limiting process

$$M(L) \to M(L_0) \text{ as } L \to L_0 \in \mathcal{N}.$$

A related operator $\mathcal{A}_L$

Our point of view is based on a *different* operator:

$$\mathcal{A}_L : \varphi \mapsto -\varphi''' - \varphi'$$

with

$$D(\mathcal{A}_L) = \{\varphi \in H^3(0, L) : \varphi(0) = \varphi(L) = 0, \varphi'(0) = \varphi'(L)\}.$$

- $\mathcal{A}_L$ is skew-adjoint⇒ good spectral properties.
- Not exactly compatible with the linearized KdV.

Two types of eigenfunctions

Consider the eigenvalues of the operator $\mathcal{A}_L$:

$$\begin{cases} f''' + f' + i\lambda f = 0, & \text{in } (0, L), \\ f(0) = f(L) = f'(0) - f'(L) = 0, \end{cases}$$

Type 1 and Type 2 eigenfucntions

- If $2k + l \notin 3\mathbb{N}^*$, $\exists! \varphi$ (**Type 1**) such that $\varphi'(0) = \varphi'(L) = 0$.
- If $2k + l \in 3\mathbb{N}^*$, solutions in the form $f = C_1\varphi + C_2\tilde{\varphi}$.
  $\tilde{\varphi}$ (**Type 2**): $\tilde{\varphi}'(0) = \tilde{\varphi}'(L) \neq 0$ and φ and $\tilde{\varphi}$ are linearly independent.

Type 1: exist for all $L \in \mathcal{N}$,

Type 2: only exist when $2k + l \in 3\mathbb{N}^*$!

Eigenmodes

The limiting problem $\mathcal{A}_L$ as $L \to L_0$: a perturbation of $\mathcal{A}_{L_0}$.

$\Rightarrow$ the asymptotic behaviors depend on the perturbation of both Type 1 and Type 2 eigenfunctions around L_0.

	$k = l$	$2k + l \notin 3\mathbb{N}^*$	$2k+l \in 3\mathbb{N}^*, k \neq l$
Eigenvalues	zero (double)	nonzero (simple)	nonzero (double)
Eigenfunctions	both Type 1 and 2	only Type 1	both Type 1 and 2
$\lvert\lambda_L - \lambda_{L_0}\rvert$	$\mathcal{O}(\lvert L - L_0\rvert)$	$\mathcal{O}(\lvert L - L_0\rvert^2)$	$\mathcal{O}(\lvert L - L_0\rvert)$
“Neumann error"	$\mathcal{O}(1)$	$\mathcal{O}(\lvert L - L_0\rvert)$	$\mathcal{O}(1)$

Inspired by these, for the classification of L, the effective factor is not $\dim M$ but the type of $2k + l \mod 3$.

Novel Classification

Let $L \in \mathcal{N}$. We say that (k, l) is an unreachable pair if

$$k^2 + kl + l^2 = 3(\frac{L}{2\pi})^2 \Leftrightarrow L = 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}}.$$

Definition (Classification of the unreachable pairs (k, l))

1. $\mathcal{S}_1(L) := \{(k, l) : k = l\}$,
2. $\mathcal{S}_2(L) := \{(k, l) : k \equiv l \mod 3,\ k \neq l\}$,
3. $\mathcal{S}_3(L) := \{(k, l) : k \not\equiv l \mod 3\}$.

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Definition (Classification of critical lengths)

1. $\mathcal{N}^1 := \{L \in \mathcal{N} : \exists!(k, l) \in \mathcal{S}_1(L)\}$,
2. $\mathcal{N}^2 := \{L \in \mathcal{N} :$ all $(k, l) \in \mathcal{S}_3(L)\}$,
3. $\mathcal{N}^3 := \{L \in \mathcal{N} : \exists(k, l) \in \mathcal{S}_2(L)\}$.

Main result

Theorem (N.–Xiang, 2025)

Let $L \in \mathcal{N}^3$. Then the system is not small-time locally null-controllable with controls in $H^{\frac{4}{3}}$ and initial and final datum in $H^4(0,L) \cap H_0^1(0,L)$.

More precisely, $\exists T_0, \varepsilon > 0$ s.t. $\forall \delta > 0$, there exists some y_0 satisfying $\|y_0\|_{H^4} < \delta$ s.t. for all control u with $\|u\|_{H^{4/3}(0,T_0)} < \varepsilon_0$, we have

$$y(T_0, \cdot) \not\equiv 0.$$

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$$y(T_0, \cdot) \not\equiv 0.$$

A complete answer to the open problem:

L	$\mathcal{N}^1$	$\mathcal{N}^2$	$\mathcal{N}^3$
Eigenfunctions	Type 1 and 2	Type 1	Type 1 and 2
$\dim M$	1	even	any integer
Small-time controllability	Positive [CC, 2003]	Negative [CKN, 2020]	Negative **Our Thm**

Outline of the presentation

Idea of proof

Our proof primarily relies on

1. Novel classification;
2. A trapping direction:
   2.1 Reduction approach;
   2.2 The remaining case under the new classification is degenerate;
   2.3 A higher-order expansion and microlocal analysis techniques.
3. Obstruction to small-time controllability.

Trapping direction

A major step is to construct a *trapping direction*.

y: solution to KdV with $y(0,\cdot) = \varepsilon\Psi(0,\cdot)$ and $u = 0$,

$$\|y(t,\cdot) - \varepsilon\Psi(t,\cdot)\|_{L^2(0,L)} \lesssim \varepsilon^2, \text{ for } t \text{ small.}$$

$\Psi(t,x)$: trapping direction

$$\begin{cases} \partial_t \Psi(t,x) + \partial_x^3 \Psi(t,x) + \partial_x \Psi(t,x) = 0, \\ \Psi(t,0) = \Psi(t,L) = \partial_x \Psi(t,0) = \partial_x \Psi(t,L) = 0. \end{cases}$$

After that, using a standard contradiction argument (with some effort...), we conclude the small-time controllability fails.

Construction of a trapping direction

Based on power series expansion. Recall in Coron–Crépeau (2003)

$$Q_M(\varphi; y) := \int_0^{\infty} \int_0^{L} |y(t,x)|^2 e^{-ipt} \varphi'(x) dx dt.$$

$$\varphi(x) = \sum_{j=1}^{3} (\eta_{j+1} - \eta_j) e^{\eta_{j+2} x} \in M$$

with $\eta_j^3 + \eta_j + ip = 0$.

- $Q_M \equiv 0$ implies $y \sim \varepsilon y_1 + \varepsilon^2 y_2$ is still not controllable.
- Further consider $y \sim \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3$: the small-time controllability.

Construction of a trapping direction

Coron–Koenig–Nguyen (2020) first view Q_M as a Fourier transform w.r.t. t.

$$Q_M = \int_0^L \mathcal{F}_{t\to p}(|y|^2(\cdot,x)\varphi'(x))dx.$$

After a direct computation,

$$Q_M = \int_0^L \int_0^\infty |y(t,x)|^2 \varphi_x(x) e^{-ipt} dt dx = \int_{\mathbb{R}} \hat{u}(\tau)\overline{\hat{u}(\tau - p)} \int_0^L B(\tau,x) dx d\tau,$$

$$B(\tau,x) = \frac{\sum_{j=1}^3 (e^{\lambda_{j+1}L} - e^{\lambda_j L}) e^{\lambda_{j+2}x}}{\sum_{j=1}^3 (\lambda_{j+1} - \lambda_j) e^{-\lambda_{j+2}L}} \cdot \frac{\sum_{j=1}^3 (e^{\tilde\lambda_{j+1}L} - e^{\tilde\lambda_j L}) e^{\tilde\lambda_{j+2}x}}{\sum_{j=1}^3 (\tilde\lambda_{j+1} - \tilde\lambda_j) e^{-\tilde\lambda_{j+2}L}} \cdot \varphi'(x).$$

- λ_j: $\lambda_j^3 + \lambda_j + i\tau = 0$, $\quad \tilde\lambda_j$: $\tilde\lambda_j^3 + \tilde\lambda_j - i(\bar\tau - p) = 0$, $j = 1, 2, 3$.

Reduction approach

For

$$Q_M = \int_{\mathbb{R}} \hat{u}(\tau)\overline{\hat{u}(\tau - p)} \int_0^L B(\tau, x) dx d\tau.$$

They introduced a *reduction approach*:

- $\int_0^L B(\tau, x) dx = \frac{E}{|\tau|^{\frac{4}{3}}} + \mathcal{O}(|\tau|^{-1})$
- Coercive estimates for Q_M:

$$\int_0^\infty \int_0^L |y(t,x)|^2 e^{-ipt} \varphi_x(x) dx dt \sim \|u\|^2_{H^{-\frac{2}{3}}} \left(E + \mathcal{O}(1)T\right).$$

- Construct $\Psi = \Re(E e^{-ipt} \varphi_x)$.

Degenerate case

Notice that $L \in \mathcal{N}^3 \Rightarrow E = 0$ in Coron–Koenig–Nguyen's approach.

$L \in \mathcal{N}^3$ is a degenerate case.

- The appearance of Type 2 eigenfunctions.
  More delicate analysis to detect the non-vanishing term at higher orders of B and Q_M.
- The regularity level is lower: $(1+|D_t|^2)^{-\frac{1}{6}}$ involves.
  More techniques in microlocal analysis $+$ specific lemmas concerning Sobolev norms on compactly supported functions.

Refined reduction approach

Our refined reduction approach:

- **Step 1:** $\int_0^L B(\tau, x)dx = \frac{E}{|\tau|^2} + \mathcal{O}(|\tau|^{-\frac{7}{3}})$.
- **Step 2:** coercive estimate of Q_M

$$\int_0^\infty \int_0^L |y(t,x)|^2 e^{-ipt} \varphi_x(x) dx dt \sim \|u\|_{H^{-1}}^2 (E + \mathcal{O}(T^{\frac{1}{100}})).$$

- **Step 3:** construct the trapping direction Ψ .

Step 1: Asymptotic analysis on B

Lemma

Let $p \in \mathbb{R}$, and let φ be defined by $\varphi(x) = \sum_{j=1}^{3} (\eta_{j+1} - \eta_j) e^{\eta_{j+2} x}$. Assume that $\eta_j \neq 0$ and moreover, $e^{\eta_j L} = 1$ and $\eta_j^3 + \eta_j + ip = 0$, for $j = 1, 2, 3$. We have

$$\int_0^L B(\tau, x) dx = \frac{E}{|\tau|^2} + O(|\tau|^{-\frac{7}{3}}),$$

where E is defined by $E = \frac{1}{27} p^2 L \sum_{j=1}^{3} \frac{\eta_{j+1} - \eta_j}{\eta_{j+2}}$.

We use

$$\lambda_j = \mu_j \tau^{\frac{1}{3}} - \frac{1}{3\mu_j} \tau^{-\frac{1}{3}} + \frac{1}{81\mu_j^5} \tau^{-\frac{5}{3}} + \mathcal{O}(\tau^{-2}), \quad |\tau| \gg 1$$

$$\tilde{\lambda}_j = \tilde{\mu}_j \bar{\tau}^{\frac{1}{3}} - \frac{1}{3\tilde{\mu}_j} \bar{\tau}^{-\frac{1}{3}} + \frac{1}{81\tilde{\mu}_j^5} \bar{\tau}^{-\frac{5}{3}} + O(\bar{\tau}^{-2}), \quad |\tau| \gg 1,$$

where $\mu_j = e^{-\frac{i\pi}{6} - \frac{2ij\pi}{3}}$ and $\tilde{\mu}_j = e^{\frac{i\pi}{6} + \frac{2ij\pi}{3}}$.

Step 2: Coercive estimates

Proposition

Let $u \in L^2(\mathbb{R}_+)$ *with* $u \not\equiv 0$*, and* $y \in C(\mathbb{R}_+; L^2(0,L)) \cap L^2_{loc}\left(\mathbb{R}_+; H^1(0,L)\right)$ *solution of KdV with* $u(t) = 0, y(t,\cdot) = 0$ *for* $t > T$*. Then,* $\exists\, N(u) \geq 0$ *s.t.* $N(u) \sim \|u\|_{H^{-1}}$

$$\int_0^\infty \int_0^L |y(t,x)|^2 e^{-ipt} \varphi_x(x)\,dx\,dt = N(u)^2 \left(E + O(1) T^{\frac{1}{100}}\right).$$

$$\begin{aligned}\int_0^\infty \int_0^L |y(t,x)|^2 e^{-ipt} \varphi_x(x)\,dx\,dt &= \int_{\mathbb{R}} \hat{u}(\tau)\overline{\hat{u}(\tau - p)} \int_0^L B(\tau, x)\,dx\,d\tau \\ &\sim \int_{\mathbb{R}} \hat{u}(\tau)\overline{\hat{u}(\tau - p)}\left(\frac{E}{|\tau|^2} + O(|\tau|^{-\frac{7}{3}})\right)d\tau\end{aligned}$$

A key step is to prove $\|\langle D_t\rangle^{-\frac{1}{3}} w\|_{L^2}^2 \sim \|w\|_{H^{-\frac{1}{3}}}^2 (1 + O(T^{\varepsilon}))$, if $\operatorname{supp} w \subset [-T, T]$.

Rough proof

A key step is to prove $\|\langle D_t\rangle^{-\frac{1}{3}} w\|_{L^2}^2 \sim \|w\|_{H^{-\frac{1}{3}}}^2 (1 + O(T^{\varepsilon}))$, if $\operatorname{supp} w \subset [-T, T]$.

- Based on complex analysis, using Palay-Werner's Theorem and several special entire functions, construct w and establish the relation between w and u.
- Split high frequency and low frequency, establish error estimates w.r.t. $\|w\|_{H^{-\frac{1}{3}}}$.
- Choosing a good cutoff size T^{β} and $\chi(\frac{t}{T^{\beta}})$ compatible with uncertainty principles.
- Using microlocal techniques to prove commutator estimates.

Step 3: Construct the trapping direction

$\Psi(t,x) = \Re(Ee^{-ipt}\varphi_x)$: trapping direction

$$\begin{cases} \partial_t \Psi(t,x) + \partial_x^3 \Psi(t,x) + \partial_x \Psi(t,x) = 0, \\ \Psi(t,0) = \Psi(t,L) = \partial_x \Psi(t,0) = \partial_x \Psi(t,L) = 0. \end{cases}$$

Then, y: solution to KdV with $y(0,\cdot) = \varepsilon \Psi(0,\cdot)$ and $u = 0$,

$$\|y(t,\cdot) - \varepsilon \Psi(t,\cdot)\|_{L^2(0,L)} \lesssim \varepsilon^2, \text{ for } t \text{ small.}$$

Further perspectives

Some related topics for nonlinear KdV

- Regularity of the control: In Coron–Koenig–Nguyen, $u \in H^{\frac{2}{3}}(\mathbb{R}_+)$; we use $u \in H^{\frac{4}{3}}(\mathbb{R}_+)$. Optimal $H^s(\mathbb{R}_+)$? What if $L^2(\mathbb{R}_+)$?
- Size of the control: $\|u\|_{H^s(\mathbb{R}_+)} < \varepsilon$. What happens if we allow big control?
- Size of initial data: $\|y^0\|_{L^2(0,L)} < \varepsilon$. Can we get global/semiglobal controllability for big data?

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Concerning our classification,

- Controllability and stability as $L \to \mathcal{N}$: $\|y_0\|^2_{L^2(0,L)} \leq C(T,L) \int_0^T |\partial_x y(t,0)|^2 dt$.

$$C(T,L) \to \infty, \text{ as } L \to \mathcal{N}. \text{ But at what rate for } T \text{ and } L?$$

- Exponential stabilization at critical lengths: open in $\mathcal{N}^3$.
- Asymptotic stability at critical lengths: open in $\mathcal{N}^3$.

Thank you for your attention!