Local Controllability of nonlinear Schrödinger equations

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Controllability

Consider $\omega \subset \mathbb{T}^d$ to be a nonempty open set. This is the basic geometric setting for the interior control problem.

$$(\mathrm{i}\partial_t - \Delta)u = f\mathbb{1}_\omega(x)\mathbb{1}_{(0,T)}(t), \tag{LSch}$$

where $f \in L^2((0,T) \times \omega)$, For this model, we say the linear Schrödinger equation is (exactly/null) controllable in time $T > 0$ if:

Exact controllability

For any initial datum $u_{in} \in L^2$ and any target $u_{end} \in L^2$, there exists $f \in L^2((0,T) \times \omega)$ such that the solution u satisfies $u|_{t=0} = u_{in}$ and $u|_{t=T} = u_{end}$.

Null controllability

For any initial datum $u_{in} \in L^2$, there exists $f \in L^2((0,T) \times \omega)$ such that the solution u satisfies $u|_{t=0} = u_{in}$ and $u|_{t=T} = 0$.

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Observability

For a homogeneous Schrödinger equation:

$$(\mathrm{i}\partial_t - \Delta)v = 0, v|_{t=0} = v_{in} \quad \text{(Ad)}$$

Definition

We say a homogeneous equation above is observable in $[0, T] \times \omega$ if there exists a constant $C > 0$ such that every solution $v \in C^0(0, T, L^2)$ of the homogeneous Schrödinger equation satisfies

$$C \int_0^T \int_\omega |v|^2 dxdt \geq ||v_{in}||_{L^2}^2. \quad \text{(Obs)}$$

Here the inequality (Obs) is called the observability inequality for the adjoint equation.

Hilbert Uniqueness Method

According to the Hilbert Uniqueness Method, the controllability is equivalent to an observability inequality for the adjoint system.

Theorem

The Schrödinger equation (LSch) *is null controllable if and only if the adjoint equation* (Ad) *is observable in* $[0, T] \times \omega$*.*

We define the operator R by

$$R : f \in L^2((0, T) \times \omega) \mapsto u_{in} \in L^2, \qquad (1)$$

where u is the solution of (LSch) with $u|_{t=T} = 0$. On the other hand, we define the operator S by

$$S : v_{in} \in L^2 \mapsto v\mathbf{1}_{(0,T)}(t)\mathbf{1}_{\omega}(x) \in L^2((0, T) \times \omega), \qquad (2)$$

where v solves the adjoint equation (Ad). Therefore, the null controllability is just the surjectivity of the operator R and the observability is just the coercivity of the operator S.

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Geometric control condition

In $\mathbb{T}^d$, The so-called geometric control condition is the following one:

Definition (GCC)

We say that a nonempty open subset $\omega \subset \mathbb{T}^d$ satisfies the geometric control condition if every geodesic of $\mathbb{T}^d$ eventually enters into ω, which means that for every $(x, \xi) \in \mathbb{T}^d \times \mathbb{S}^{d-1}$, there exists some $t \in (0, \infty)$, such that $x + t\xi \in \omega$.

General Scheme for Linear problems

To prove the observability, we use the semiclassical approach. We cut off the initial data near the frequency of size h^{-1}. Then analyze the propagation of the semiclassical equation:

$$\mathrm{i}h\partial_s v_h - h^2\Delta v_h = f_h, \tag{3}$$

with $v_h|_{t=0} = v_h^0 = \varphi(hD)v_{in}$ and $f_h \in L^2([0,T]\times\mathbb{T}^d)$. For this semiclassical equation, we have the weak observability

$$||v_h^0||_{L^2}^2 \lesssim \int_0^T \int_\omega |v_h|^2 dxdt + h^{-2}\|f_h\|_{L^2}^2 \tag{WObs}$$

To derive the observability, we sum up the inequality for $h = 2^j$ (Littlewood-Palay decomposition).

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General setting

In this sequel, we consider the linear/nonlinear Schrödinger equations on tori $\mathbb{T}^d$. The associated GCC is the following one:

Definition (GCC)

We say that a nonempty open subset $\omega \subset \mathbb{T}^d$ satisfies the geometric control condition if every geodesic of $\mathbb{T}^d$ eventually enters into ω, which means that for every $(x, \xi) \in \mathbb{T}^d \times \mathbb{S}^{d-1}$, there exists some $t \in (0, \infty)$, such that $x + t\xi \in \omega$.

We aim to prove the exact controllability for the following quasilinear Schrödinger equation:

$$\mathrm{i}u_t + \Delta u + g_1'(|u|^2)\Delta\big(g_1(|u|^2)\big)u + g_2(|u|^2)u = f \quad (t, x) \in [0, T] \times \mathbb{T}^d, \tag{4}$$

where g_1 and g_2 are polynomial functions of degree ≥ 1, vanishing at the origin.

Main result

Theorem

Let $T > 0$, and suppose that $\omega \subset \mathbb{T}^d$ satisfies GCC. For any $s > d/2 + 2$, $\exists \epsilon_0 > 0$ sufficiently small s.t. for $\forall u_{in}, u_{end} \in H^s$ satisfying $\|u_{in}\|_{H^s} + \|u_{end}\|_{H^s} < \epsilon_0$ the following holds. $\exists f \in C([0, T], H^s)$ with $\operatorname{supp} f(t, \cdot) \subset \omega$ for any $t \in [0, T]$ and there exists a unique solution $u \in C([0, T], H^s)$ of

$$\begin{cases} \mathrm{i}u_t + \Delta u + g_1'(|u|^2)\Delta\big(g_1(|u|^2)\big)u + g_2(|u|^2)u = f, \\ u\big|_{t=0} = u_{in}, \end{cases} \tag{NLS}$$

which verifies that $u\big|_{t=T} = u_{end}$.

Choice of control function

Theorem

Under same assumptions, for all $u_{in} \in H^s$ *satisfying* $\|u_{in}\|_{H^s} < \epsilon_0$ *the following holds.* $\exists \tilde{f} \in C([0,T], H^s)$ *and there exists a unique solution* $u \in C([0,T], H^s)$ *of*

$$\begin{cases} \mathrm{i}u_t + \Delta u + g_1'(|u|^2)\Delta\big(g_1(|u|^2)\big)u + g_2(|u|^2)u = \chi_T \varphi_\omega \tilde{f}, \\ u|_{t=0} = u_{in}, \end{cases} \tag{5}$$

which verifies that $u|_{t=T} = 0$*. Furthermore, the control function verifies*

1. $\tilde{f} \in C([0,T], H^s)$*;*
2. $\chi_T(\cdot) = \chi_1(\cdot/T) \in C^\infty(\mathbb{R})$ *where* $\chi_1(t) = 1$ *for* $t \leq \frac{1}{2}$ *and* $\chi_1(t) = 0$ *for* $t \geq \frac{3}{4}$*;*
3. $0 \leq \varphi_\omega \in C^\infty(\mathbb{T}^d)$ *satisfies* $\mathbb{1}_{\omega'} \leq \varphi_\omega \leq \mathbb{1}_\omega$*, where* ω' *also satisfies the geometric control condition and* $\overline{\omega'} \subset \omega$*. Such* ω' *exists because that* $\mathbb{T}^d$ *is compact.*

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Paralinearization

The first step is to consider a sequence of linear problems that approximates the non-linear one. Let

$$U := \begin{bmatrix} u \\ \bar{u} \end{bmatrix}, \quad F := \begin{bmatrix} f \\ \bar{f} \end{bmatrix}, \quad E := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbb{1} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using paraproduct, we could write (NLS) into the following form:

$$\partial_t U = \mathrm{i}\mathcal{A}(U)U + R(U)U - \mathrm{i}\chi_T \varphi_\omega EF, \tag{NParaEq}$$

where $\mathcal{A}$ is a self-adjoint para-differential operator. So we define the iterative scheme as follows. Set $U^0 = 0$, $F^0 = 0$, and $(U^{n+1}, F^{n+1}) \in (C([0,T], H^s))^2$ by letting $F^{n+1} = \mathcal{L}_I(U^n)U_{in}$ and U^{n+1} solve

$$\begin{cases} \partial_t U^{n+1} = \mathrm{i}\mathcal{A}(U^n)U^{n+1} + R(U^n)U^{n+1} - \mathrm{i}\chi_T \varphi_\omega EF^{n+1}, \\ U^{n+1}(0) = U_{in}, U^{n+1}(T) = 0. \end{cases}$$

Linear controllability

We consider the controllability of the linear problem: $\partial_t U = \mathrm{i}\mathcal{A}(\underline{U})U - \mathrm{i}\chi_T\varphi_\omega EF$ and define the control operator

$$\mathscr{L}(\underline{U}) : H^s \to C([0,T], H^s), \tag{6}$$

for any $U_{in} \in H^s$,

$$F = \mathscr{L}(\underline{U})U_{in} \in C([0,T], H^s)$$

sends the initial datum $U(0) = U_{in}$ to the final target $U(T) = 0$. Then we need to prove the observability for adjoint system $\partial_t V = -\mathrm{i}\mathcal{A}(\underline{U})V$.

$$||V(0)||_{L^2}^2 \leq C \int_0^T ||\chi_T\varphi_\omega V(t)||_{L^2}^2 dt.$$

1. Using the similar compactness-uniqueness method, high-frequency estimates by semi-classical defect measure, and low frequency by unique continuation.
2. Verify that $\mathscr{L}(\underline{U}) : H^s \to C([0,T], H^s)$ using energy estimates

Lemma

Suppose that $s > \frac{d}{2} + 2$, then for ϵ_0 sufficiently small,

$$\|\mathcal{L}_I(\underline{U}_1) - \mathcal{L}_I(\underline{U}_2)\|_{\mathscr{L}(H^s, C([0,T],H^{s-2}))} \lesssim \|\underline{U}_1 - \underline{U}_2\|_{L^\infty([0,T],H^{s-2})}.$$

Based on the lemma, we prove the convergence of the sequence (U^n, F^n), which implies the null controllability. To recover the exact controllability, we use the time-reversed equation.

Some comments

- What happens if we have no GCC?
- What happens if we are not in a flat geometry? With boundary?

Waveguide setting: $\mathbb{R}^2 \times \mathbb{T}$

Geometric setting

$\Omega = (\Omega_1, \Omega_2) \subset \mathbb{R}^2 \times \mathbb{T}$. Let $\Omega_1 \subset \mathbb{R}^2$ be a nonempty, open, $2\pi\mathbb{Z}^2$-invariant set. Let $\Omega_2 \subset \mathbb{T}$ be open and nonempty.

We consider the local controllability for

$$\begin{cases} \mathrm{i}\partial_t u + \Delta_{x,y} u \pm |u|^2 u = f & \text{on } [0, T] \times \mathbb{R}^2 \times \mathbb{T}, \\ u|_{t=0} = u_0 & \text{on } \mathbb{R}^2 \times \mathbb{T}, \end{cases} \tag{7}$$

Theorem

This system can fulfill local controllability.

Thank you for your attention!